

CONDITIONAL IDENTITIES

Section - 6

6.1

TYPE-I

Problems based on transformation of the plus form of sine or cosine in product form :

Working Rule :

- (i) Simplify the terms containing A and B using the formula of $\sin C \pm \sin D$ and $\cos C \pm \cos D$ whichever is applicable.
- (ii) Simplify the term containing C by using $\sin 2\theta = 2 \sin \theta \cos \theta$ or $\cos 2\theta = 2 \cos^2 \theta - 1$ or $\cos 2\theta = 1 - 2 \sin^2 \theta$, and write $A + B$ in terms of C and then take out the common factor (which will be a term containing C).
- (iii) Then transform C in $A + B$ (not in the factor which has been taken common).
- (iv) Then simplify to get the desired result.

Illustrating the Concepts :

If $A + B + C = \pi$, then show that :

- (i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
- (ii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$
- (iii) $\cos A + \cos B + \cos C = 1 + 4 \sin A/2 \sin B/2 \sin C/2$
- (iv) $\sin A + \sin B + \sin C = 4 \cos A/2 \cos B/2 \cos C/2$

$$(i) \quad \text{LHS} = \sin 2A + \sin 2B + \sin 2C$$

$$= 2 \sin (A + B) \cos (A - B) + \sin 2C$$

$$= 2 \sin (\pi - C) \cos (A - B) + \sin 2C$$

[Convert $A + B$ to $\pi - C$]

$$= 2 \sin C \cos (A - B) + 2 \sin C \cos C$$

$$= 2 \sin C [\cos (A - B) + \cos C]$$

[Take the term involving C as factor]

Convert the expression with-in brackets in A and B only.

$$= 2 \sin C [\cos (A - B) + \cos (\pi - A - B)]$$

$$= 2 \sin C [\cos (A - B) - \cos (A + B)]$$

$$= 2 \sin C (2 \sin A \sin B)$$

$$= 4 \sin A \sin B \sin C = \text{RHS}$$

$$(ii) \quad \text{LHS} = \cos 2A + \cos 2B + \cos 2C$$

$$= 2 \cos (A + B) \cos (A - B) + \cos 2C$$

$$= -2 \cos C \cos (A - B) + 2 \cos^2 C - 1$$

$$= -2 \cos C [\cos (A - B) - \cos C] - 1$$

$$= -2 \cos C [\cos (A - B) + \cos (A + B)] - 1$$

$$= -2 \cos C [2 \cos A \cos B] - 1$$

$$= -1 - 4 \cos A \cos B \cos C = \text{RHS}$$

$$\begin{aligned}
 \text{(iii)} \quad \text{LHS} &= \cos A + \cos B + \cos C \\
 &= 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} + \cos C = 2 \cos \left(\frac{\pi}{2} - \frac{C}{2} \right) \cos \frac{A-B}{2} + \cos C \\
 &= 2 \sin \frac{C}{2} \cos \frac{A-B}{2} + \left(1 - 2 \sin^2 \frac{C}{2} \right) \quad \left[\text{Express } \cos C \text{ in terms of } \sin \frac{C}{2} \right] \\
 &= 2 \sin \frac{C}{2} \left[\cos \frac{A-B}{2} - \sin \frac{C}{2} \right] + 1 = 2 \sin \frac{C}{2} \left[\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right] + 1 \\
 &= 2 \sin \frac{C}{2} \left[2 \sin \frac{A}{2} \sin \frac{B}{2} \right] + 1 = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \text{RHS}
 \end{aligned}$$

(iv) Try On The Same Pattern. (As in part 3 above)

6.2

TYPE-II

Problems involving squares of sines or cosines of angles.

Working Rule :

- (i) First, rearrange the terms in L.H.S. so that either $\sin^2 A - \sin^2 B = \sin(A+B) \cdot \sin(A-B)$ or $\cos^2 A - \sin^2 B = \cos(A+B) \cdot \cos(A-B)$ can be applied.
- (ii) Simplify the term containing C and then take out the common factor (which will be a term containing C).
- (iii) Then transform C in $A+B$ (not in the factor which has been taken common).
- (iv) Then simplify to get the desired result.

Illustrating the Concepts :

If $A + B + C = \pi$, then show that :

- (i) $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$
- (ii) $\cos^2 A/2 + \cos^2 B/2 + \cos^2 C/2 = 2 + 2 \sin A/2 \sin B/2 \sin C/2$
- (iii) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$
- (i) Starting from L.H.S. :

$$\begin{aligned}
 &= \sin^2 A + \sin^2 B - \sin^2 C \\
 &= \sin^2 A + \sin(B+C) \sin(B-C) \\
 &= \sin^2 A + \sin(\pi - A) \sin(B-C) \\
 &= \sin A [\sin A + \sin(B-C)] \\
 &= \sin A \{ \sin[\pi - (B+C)] + \sin(B-C) \} \\
 &= \sin A [\sin(B+C) + \sin(B-C)] \\
 &= \sin A [2 \sin B \cos C] = 2 \sin A \sin B \cos C
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \text{L.H.S.} &= \cos^2 A/2 + (1 - \sin^2 B/2) + \cos^2 C/2 \\
 &= 1 + (\cos^2 A/2 - \sin^2 B/2) + \cos^2 C/2 = 1 + \cos(A+B)/2 \cos(A-B)/2 + \cos^2 C/2 \\
 &= 1 + \sin C/2 \cos(A-B)/2 + 1 - \sin^2 C/2 = 2 + \sin C/2 [\cos(A-B)/2 - \sin C/2] \\
 &= 2 + \sin C/2 [\cos(A-B)/2 - \cos(A+B)/2] = 2 + 2 \sin C/2 \sin A/2 \sin B/2
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad \text{L.H.S.} &= \sin^2 A + \sin^2 B + \sin^2 C \\
 &= 1 - (\cos^2 A - \sin^2 B) + \sin^2 C = 1 - \cos(A+B) \cos(A-B) + \sin^2 C \\
 &= 1 + \cos C \cos(A-B) + 1 - \cos^2 C = 2 + \cos C [\cos(A-B) - \cos C] \\
 &= 2 + \cos C [\cos(A-B) + \cos(A+B)] = 2 + 2 \cos C \cos A \cos B
 \end{aligned}$$

6.3

TYPE III

Problems involving only tangents or cotangents of angles.

Working Rule :

- (i) Write the given relation between angles keeping two angles on one side and remaining on the other side.
- (ii) Take tangent or cotangent of both sides and simplify to get the desired result.

Illustrating the Concepts :

(i) In a $\triangle ABC$, prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

(ii) If $A + B + C = \pi$, prove that : $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$.

(iii) If $A + B + C = \pi$, prove that $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$.

(i) In $\triangle ABC$, $\therefore A + B + C = \pi \quad \therefore A + B = \pi - C$

$$\Rightarrow \tan(A+B) = \tan(\pi - C) \quad [\text{By taking tan on both side}]$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C \quad [\because \tan(\pi - C) = -\tan C]$$

$$\Rightarrow \tan A + \tan B = -\tan C + \tan A \tan B \tan C$$

$$\Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

(ii) $\therefore A + B + C = \pi$

$$\therefore \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} \quad \text{or,} \quad \frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$$

$$\therefore \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot \frac{C}{2} \quad [\text{By taking tan on both side}]$$

$$\Rightarrow \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{1 - \tan \frac{A}{2} \tan \frac{B}{2}} = \frac{1}{\tan \frac{C}{2}}$$

$$\Rightarrow \tan \frac{A}{2} \tan \frac{C}{2} + \tan \frac{B}{2} \tan \frac{C}{2} = 1 - \tan \frac{A}{2} \tan \frac{B}{2} \Rightarrow \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1.$$

$$\begin{aligned}
 \text{(iii)} \quad & \therefore A + B + C = \pi \quad \therefore A + B = \pi - C \\
 & \Rightarrow \cot(A + B) = \cot(\pi - C) \quad [\text{By taking cot on both side}] \\
 & \Rightarrow \frac{\cot A \cdot \cot B - 1}{\cot B + \cot A} = -\cot C \\
 & \Rightarrow \cot A \cot B - 1 = -\cot B \cot C - \cot C \cot A \\
 & \Rightarrow \cot A \cot B + \cot B \cot C + \cot C \cot A = 1
 \end{aligned}$$

6.4

TYPE-IV

Miscellaneous Type

Working Rule :

- (i) If it is given that $x + y + z = xyz$ or, $xy + yz + zx = 1$ put $x = \tan A$, $y = \tan B$ and $z = \tan C$ in it and simplify it to find the value of $A + B + C$.
- (ii) Then put $\tan A$, $\tan B$, $\tan C$ in place of x , y , z respectively in the L.H.S. of the given question and simplify it to get the desired identify using the value of $A + B + C$.

Illustrating the Concepts :

If $xy + yz + zx = 1$, prove that $\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} = \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$

Let $x = \tan A$, $y = \tan B$, $z = \tan C$

Given, $xy + yz + zx = 1$

$$\therefore \tan A \tan B + \tan B \tan C + \tan C \tan A = 1$$

$$\Rightarrow \tan C (\tan A + \tan B) = 1 - \tan A \tan B$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{1}{\tan C} = \cot C = \tan\left(\frac{\pi}{2} - C\right)$$

$$\Rightarrow \tan(A + B) = \tan\left(\frac{\pi}{2} - C\right)$$

$$\therefore A + B = \frac{\pi}{2} - C \quad [\text{Taking principal value}]$$

$$\Rightarrow A + B + C = \frac{\pi}{2} \quad \dots \text{(i)}$$

Now, L.H.S. = $\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2}$

$$= \frac{\tan A}{1 - \tan^2 A} + \frac{\tan B}{1 - \tan^2 B} + \frac{\tan C}{1 - \tan^2 C} = \frac{1}{2} \left[\frac{2 \tan A}{1 - \tan^2 A} + \frac{2 \tan B}{1 - \tan^2 B} + \frac{2 \tan C}{1 - \tan^2 C} \right]$$

$$= \frac{1}{2} (\tan 2A + \tan 2B + \tan 2C) \quad \dots \text{(ii)}$$

$$\text{Now from (i), } A + B + C = \frac{\pi}{2} \quad \Rightarrow \quad 2A + 2B + 2C = \pi$$

$$\text{or, } 2A + 2B = \pi - 2C \quad \Rightarrow \quad \tan(2A + 2B) = \tan(\pi - 2C)$$

$$\text{or, } \frac{\tan 2A + \tan 2B}{1 - \tan 2A \tan 2B} = -\tan 2C \quad \text{or, } \tan 2A + \tan 2B = -\tan 2C + \tan 2A \tan 2B \tan 2C$$

$$\text{or, } \tan 2A + \tan 2C + \tan 2B = \tan 2A \tan 2B \tan 2C. \quad \dots \text{(iii)}$$

$$\text{From (ii), L.H.S.} = \frac{1}{2} (\tan 2A + \tan 2B + \tan 2C)$$

$$= \frac{1}{2} (\tan 2A \tan 2B \tan 2C) \quad [\text{From (iii)}]$$

$$= \frac{1}{2} \cdot \frac{2 \tan A}{1 - \tan^2 A} \cdot \frac{2 \tan B}{1 - \tan^2 B} \cdot \frac{2 \tan C}{1 - \tan^2 C} = \frac{1}{2} \cdot \frac{2x}{1 - x^2} \cdot \frac{2y}{1 - y^2} \cdot \frac{2z}{1 - z^2} = \frac{4xyz}{(1 - x^2)(1 - y^2)(1 - z^2)}$$

Illustration - 35 If $\alpha + \beta + \gamma = \pi$, then : $\tan(\beta + \gamma - \alpha) + \tan(\gamma + \alpha - \beta) + \tan(\alpha + \beta - \gamma)$ is :

- (A) $\tan(\beta + \gamma - \alpha) \cdot \tan(\gamma + \alpha - \beta) \cdot \tan(\alpha + \beta - \gamma)$ (B) $\cot(\beta + \gamma - \alpha) \cdot \cot(\alpha + \gamma - \beta) \cdot \cot(\alpha + \beta - \gamma)$
 (C) $\cot(\beta + \gamma - \alpha) \cdot \cot(\alpha + \gamma - \beta) \cdot \tan(\alpha + \beta - \gamma)$ (D) None of these

SOLUTION : (A)

$$\text{Let } \beta + \gamma - \alpha = A, \quad \gamma + \alpha - \beta = B, \quad \text{and} \quad \alpha + \beta - \gamma = C$$

$$\text{Now, } A + B + C = \beta + \gamma - \alpha + \gamma + \alpha - \beta + \alpha + \beta - \gamma$$

$$\text{or, } A + B + C = \alpha + \beta + \gamma = \pi \quad [\text{As } \alpha + \beta + \gamma = \pi]$$

$$\text{or, } A + B = \pi - C$$

$$\therefore \tan(A + B) = \tan(\pi - C)$$

$$\text{or, } \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C \quad [\text{As } \tan(\pi - C) = -\tan C]$$

$$\text{or, } \tan A + \tan B = -\tan C + \tan A \tan B \tan C \quad \text{or, } \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

Putting the values of A , B and C , we get :

$$\tan(\beta + \gamma - \alpha) + \tan(\gamma + \alpha - \beta) + \tan(\alpha + \beta - \gamma) = \tan(\beta + \gamma - \alpha) \tan(\gamma + \alpha - \beta) \tan(\alpha + \beta - \gamma).$$